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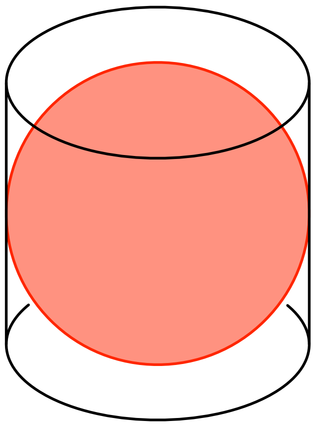
# Sphere and cylinder

## 1 The original problem

In (Richeson 2017, p 23) I found:

> "Just noticed that the ratio of the volume of a sphere to the volume of the cylinder containing it is 2:3.
> Likewise, the surface area. Eureka!"



Fir. 1: Sphere and cylinder

## 2 In other dimensions

The mentioned statement holds similar in other dimensions.
We use the following notations:

$V_{n,S}$ = Volume of the sphere in the $n$-dimensional space

$S_{n,S}$ = Surface of the sphere in the $n$-dimensional space

$V_{n,C}$ = Volume of the cylinder in the $n$-dimensional space

$S_{n,C}$ = Surface of the cylinder in the $n$-dimensional space

Examples:

| $n$ | $V_{n,S}$ | $V_{n,C}$ | Ratio | $S_{n,S}$ | $S_{n,C}$ | Ratio |
|---|---|---|---|---|---|---|
| 2 | $\pi r^2$ | $4r^2$ | $\pi:4$ | $2\pi r$ | $8r$ | $\pi:4$ |
| 3 | $\frac{4}{3}\pi r^3$ | $2\pi r^3$ | $2:3$ | $4\pi r^2$ | $6\pi r^2$ | $2:3$ |
| 4 | $\frac{1}{2}\pi^2 r^4$ | $\frac{8}{3}\pi r^4$ | $3\pi:16$ | $2\pi^2 r^3$ | $\frac{32}{3}\pi r^3$ | $3\pi:16$ |
| 5 | $\frac{8}{15}\pi^2 r^5$ | $\pi^2 r^5$ | $8:15$ | $\frac{8}{3}\pi^2 r^4$ | $5\pi^2 r^4$ | $8:15$ |
| 6 | $\frac{1}{6}\pi^3 r^6$ | $\frac{16}{15}\pi^2 r^6$ | $5\pi:32$ | $\pi^3 r^5$ | $\frac{32}{5}\pi^2 r^5$ | $5\pi:32$ |
| 7 | $\frac{16}{105}\pi^3 r^7$ | $\frac{1}{3}\pi^3 r^7$ | $16:35$ | $\frac{16}{15}\pi^3 r^6$ | $\frac{7}{3}\pi^3 r^6$ | $16:35$ |

**Tab. 1: Examples**

In every dimension there is the same ratio. In even dimensions the ratio is irrational.

## 3 General case

Notation:

$$V_{n,S}(r) = a_n r^n \tag{1}$$

Hence we get:

$$S_{n,S}(r) = \tfrac{\mathrm{d}}{\mathrm{d}r} V_{n,S}(r) = n a_n r^{n-1} \tag{2}$$

And for the volume of the cylinder:

$$V_{n,C}(r) = 2r V_{n-1,S}(r) = 2r a_{n-1} r^{n-1} = 2 a_{n-1} r^n \tag{3}$$

For the surface of the cylinder we get:

$$S_{n,C}(r) = 2V_{n-1,S}(r) + 2r S_{n-1,S}(r) = 2a_{n-1} r^{n-1} + 2r(n-1) a_{n-1} r^{n-2} = 2n a_{n-1} r^{n-1} \tag{4}$$

Remark:

$$S_{n,C}(r) = \tfrac{\mathrm{d}}{\mathrm{d}r} V_{n,C}(r) \tag{5}$$

The volumes of the sphere and the cylinder have the ratio:

$$V_{n,S}(r) : V_{n,C}(r) = a_n r^n : 2a_{n-1} r^n = a_n : 2a_{n-1} \tag{6}$$

For the surfaces we get the ratio:

$$S_{n,S}(r) : S_{n,C}(r) = n a_n r^{n-1} : 2n a_{n-1} r^{n-1} = a_n : 2a_{n-1} \tag{7}$$

From (6) and (7) we see that the ratios are equal in any dimension.

## 4 Explicit formulas

According to [1] we have:

$$a_{2k} = \tfrac{\pi^k}{k!}, \quad a_{2k+1} = \tfrac{2k!(4\pi)^k}{(2k+1)!} \tag{8}$$

From (6) and (8) we get:
In even dimensions $n = 2k$ we have:

$$\text{ratio} = \pi(2k-1)!! : 2^{k+1} k! \tag{9}$$

!! denotes the *double faktorial*, defined for odd integers $2k-1$ by:

$$(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1) = \prod_{i=1}^{k} (2i-1) = \frac{(2k)!}{2^k k!} \tag{10}$$

In odd dimensions $n = 2k+1$ we get:

$$\text{ratio} = k! 2^k : (2k+1)!! \tag{11}$$

## References

Richeson, David (2017): A-Tweeting We Will Go. Building a Professional Network with Twitter. MAA FOCUS | JUNE/JULY 2017 | maa.org/focus. 22-25.

## Websites

[1] Wikipedia: *n-sphere*
https://en.wikipedia.org/wiki/N-sphere#Volume_and_surface_area