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Regular polygon in the square

We inscribe a regular polygon in the square such that one side of the polygon is parallel to a diagonal of the square. Figure 1 depicts the situation for $n = 11$.

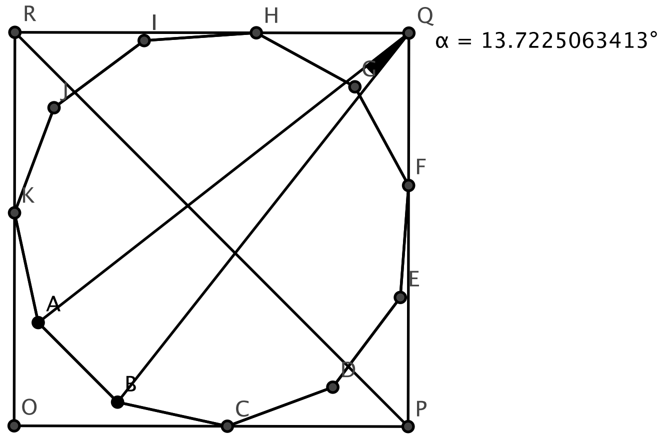

Fig. 1: Eleven-gon in the square

The angle $\alpha = \angle AQB \approx 13.7225°$ is close to 150°/11 = 13.6364°.

One may think that for $n \to \infty$ we get the limit:

$$\lim_{n\to\infty} n\alpha = 150°$$

That's not right. To see this, we consider the special case of regular n-gons where n is a multiple of 8 (Fig. 2 for $n = 24$).

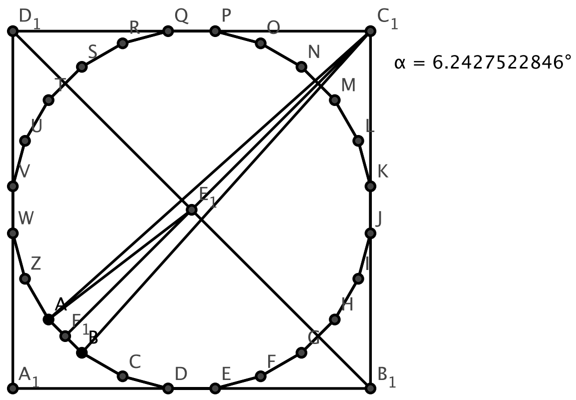

Fig. 2: 24-gon in the square

In this special case it's easy to inscribe the polygon in the square. For the angle α we have in this special case:

$$\alpha = 2\arctan\left(\frac{\tan\left(\frac{180°}{n}\right)}{1+\sqrt{2}}\right)$$

Table 1 gives some examples.

n	α	$n\alpha$
8	19.47122064°	155.7697651°
16	9.420172938°	150.7227670°
24	6.242752283°	149.8260548°
32	4.672340688°	149.5149020°
40	3.734283802°	149.3713521°
48	3.110281217°	149.2934984°
56	2.665117811°	149.2465974°
64	2.331502714°	149.2161737°
72	2.072157264°	149.1953230°
80	1.864755156°	149.1804125°
88	1.695106618°	149.1693824°
96	1.553760358°	149.1609944°
104	1.434177567°	149.1544670°
112	1.331690076°	149.1492885°
120	1.242875923°	149.1451108°
128	1.165169469°	149.1416920°
136	1.096609254°	149.1388586°
144	1.035670029°	149.1364842°
152	0.9811478612°	149.1344749°
160	0.9320797475°	149.1327596°
1000	0.1491172888°	149.1172888°
1000000	0.0001491168825°	149.1168825°

Tab. 1: Examples

We see, that the limit of $n\alpha$ seems not to be 150°.

In fact, the limit is in our case:

$$\lim_{n\to\infty} n\alpha = \lim_{n\to\infty} 2n\arctan\left(\frac{\tan\left(\frac{180^\circ}{n}\right)}{1+\sqrt{2}}\right) = \frac{360^\circ}{1+\sqrt{2}} \approx 149.1168824543124^\circ$$

We can prove this using the rule of Bernoulli – de l'Hôpital:

$$\lim_{n\to\infty} 2n\arctan\left(\lambda\tan\left(\tfrac{\pi}{n}\right)\right) = 2\lim_{m\to 0}\tfrac{1}{m}\arctan\left(\lambda\tan(m\pi)\right)$$

$$= 2\lim_{m\to 0}\frac{1}{1+\lambda^2\tan^2(m\pi)}\lambda\left(1+\tan^2(m\pi)\right)\pi = 2\lambda\pi$$

In our case we have $\lambda = \frac{1}{1+\sqrt{2}}$.