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## Parabola and right triangle

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### 1 About

We discuss a theorem connecting an arbitrary right triangle to a parabola.

### 2 Situation

Let  $ABC$  be a right triangle with right angle at  $C$  (Fig. 1).  $M$  is the center of the circumcircle  $c$  and  $F$  the foot of the perpendicular from  $C$  to the hypotenuse  $AB$ . The line  $d$  is the tangent at the circumcircle  $c$  in the point  $C$ . Of course, the segment  $MC$  is orthogonal to the tangent  $c$ .

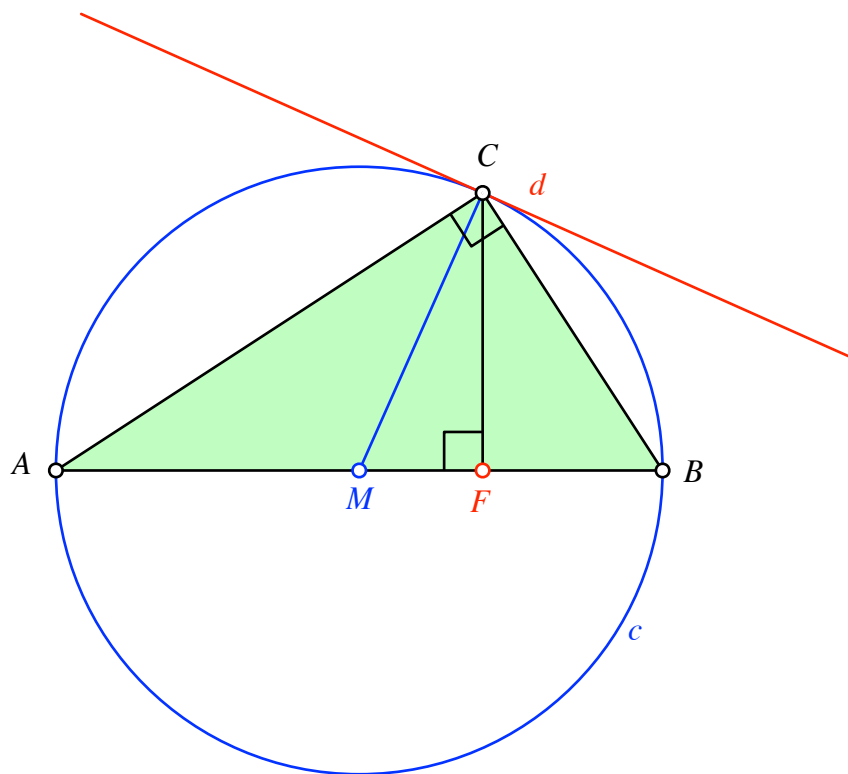
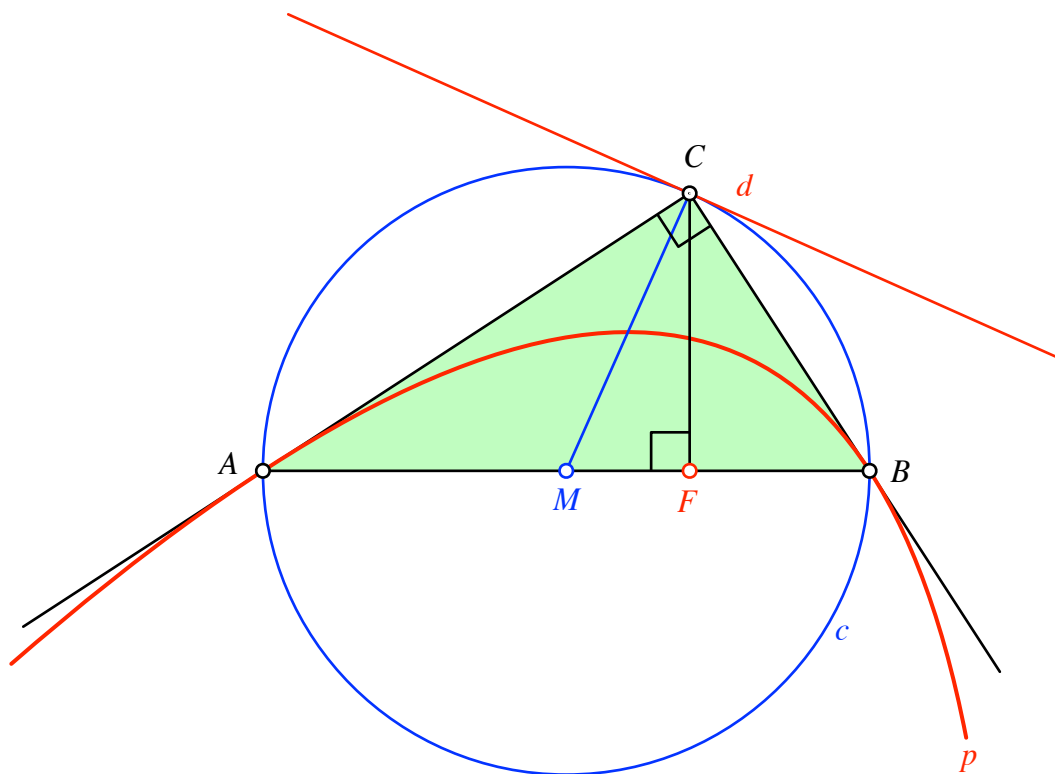


Fig. 1: Right triangle

### 3 The theorem



**Fig. 2: Parabola**

The parabola  $p$  with focus  $F$  and directrix  $d$  passes through the points  $A$  and  $B$  and is tangential to the legs  $CA$  and  $CB$  of the triangle  $ABC$  (Fig. 2).

4 Proof

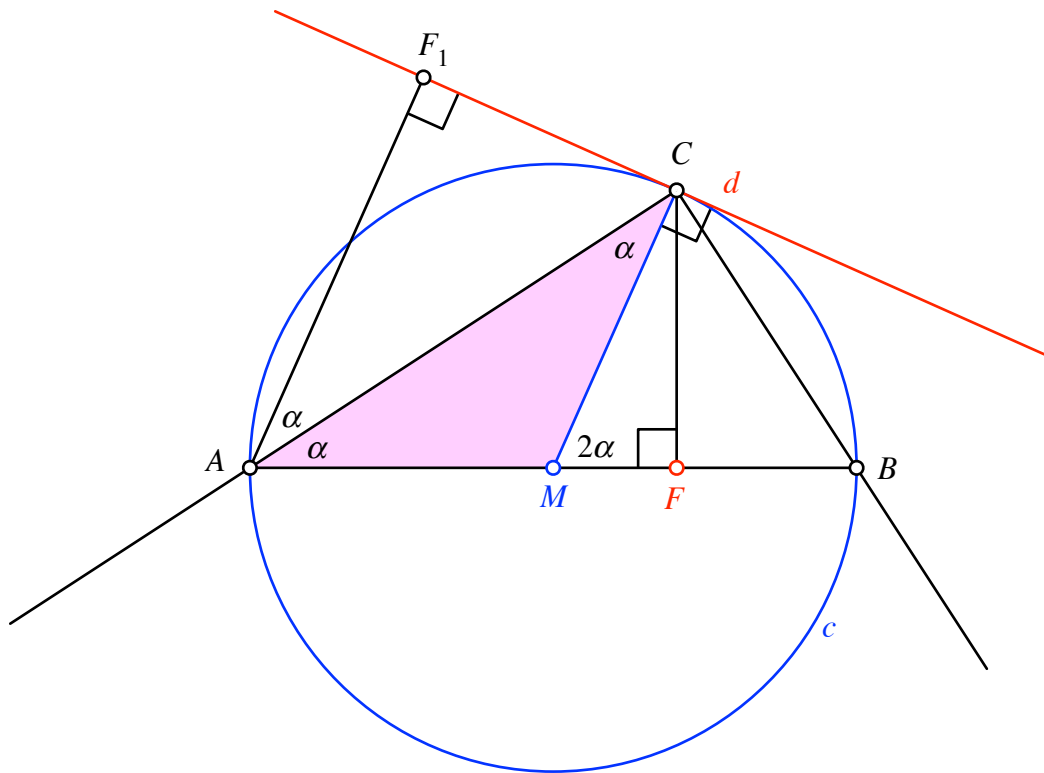
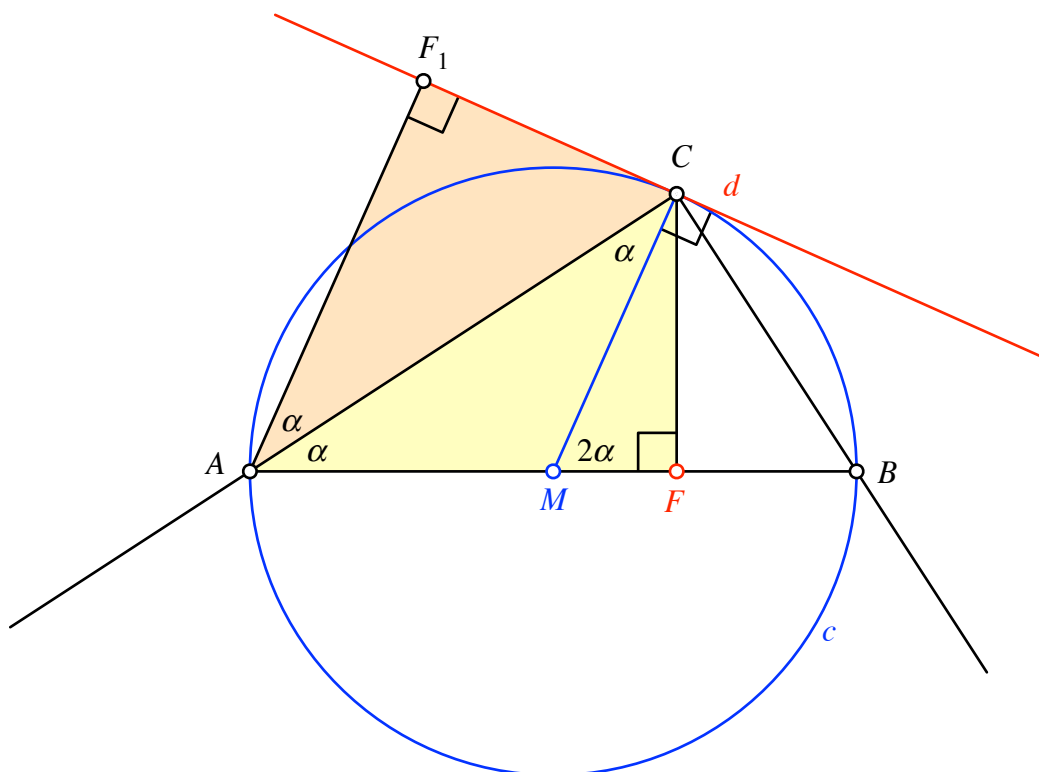


Fig. 3: Angles

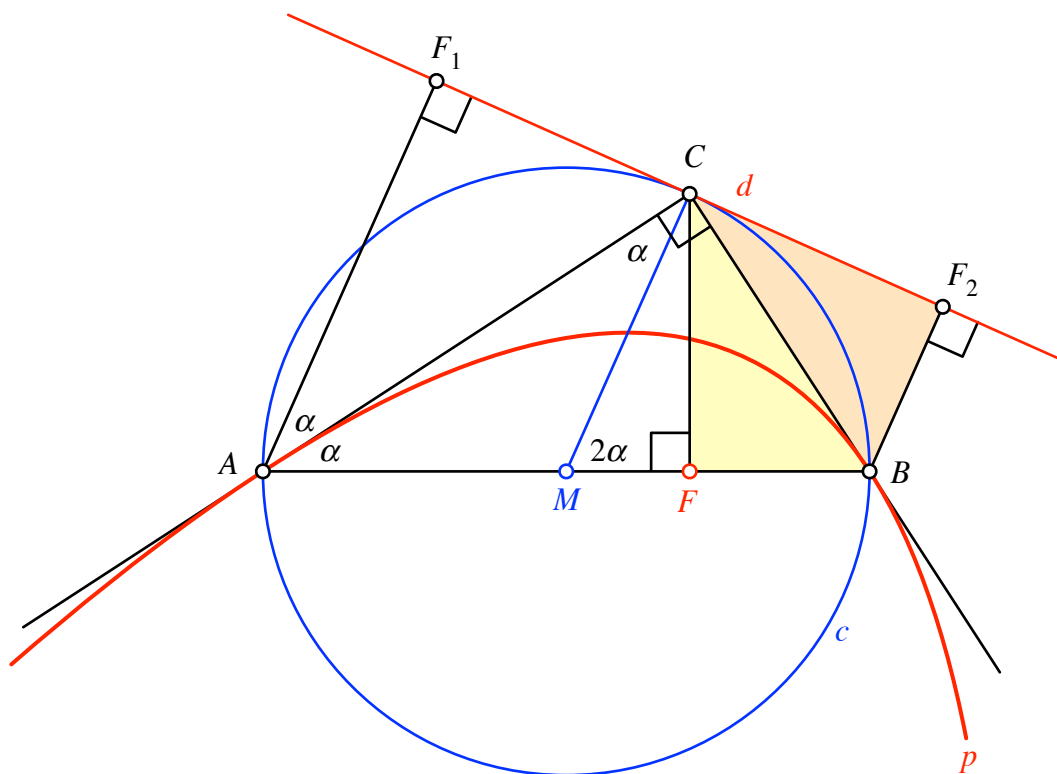
The triangle  $CAM$  is isosceles. Therefore we have the angle  $\angle BMC = 2\alpha$ . Now we draw the perpendicular from  $A$  to  $d$ . The perpendiculars  $AF_1$  and  $MC$  are parallel. Hence  $\angle BAF_1 = 2\alpha$  and  $\angle CAF_1 = \alpha$ .



**Fig. 4: Congruent triangles**

Now the triangles  $CAF$  and  $CAF_1$  are congruent (same angles, side  $CA$  in common) (Fig. 4). The triangle  $CAF_1$  is just the mirror image of the triangle  $CAF$ .

Therefore the distance from  $A$  to  $F$  is equal to the distance from  $A$  to the line  $d$ . Hence  $A$  is a point of the parabola  $p$ . The line  $AC$  is angle bisector of the angle  $\angle FAF_1$  and therefore it is tangential to the parabola  $p$ .

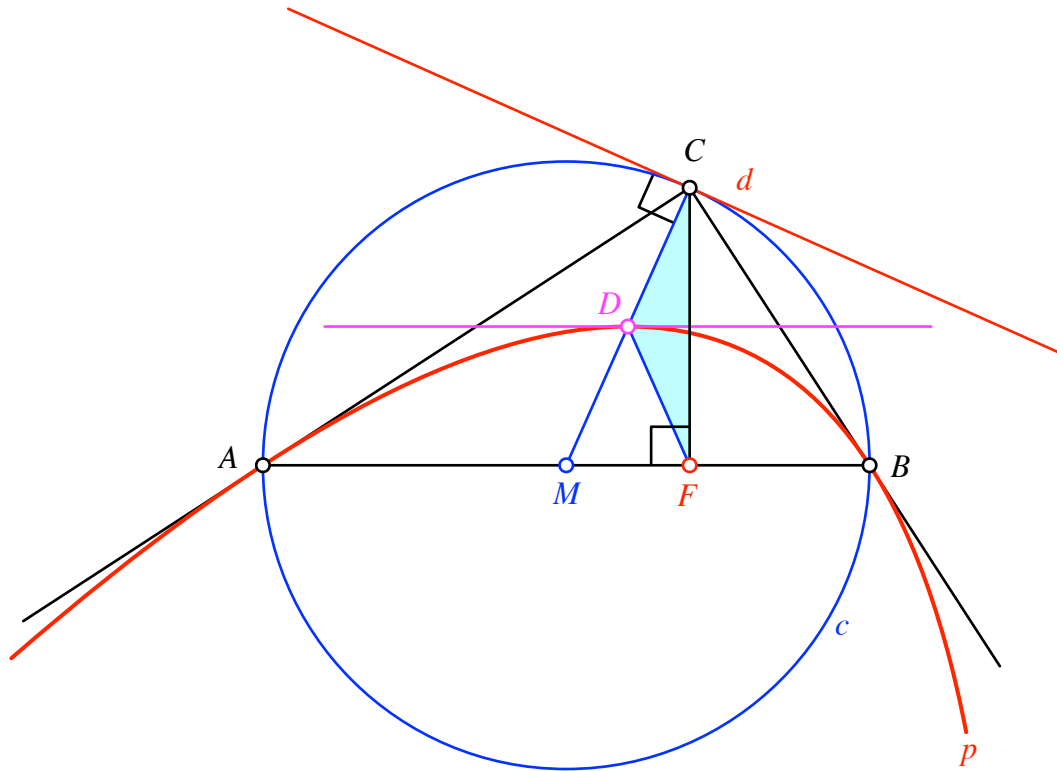


**Fig. 5: Argumentation on the other side**

An analogue argumentation holds for the point  $B$  (Fig. 5).

## 5 More tangents

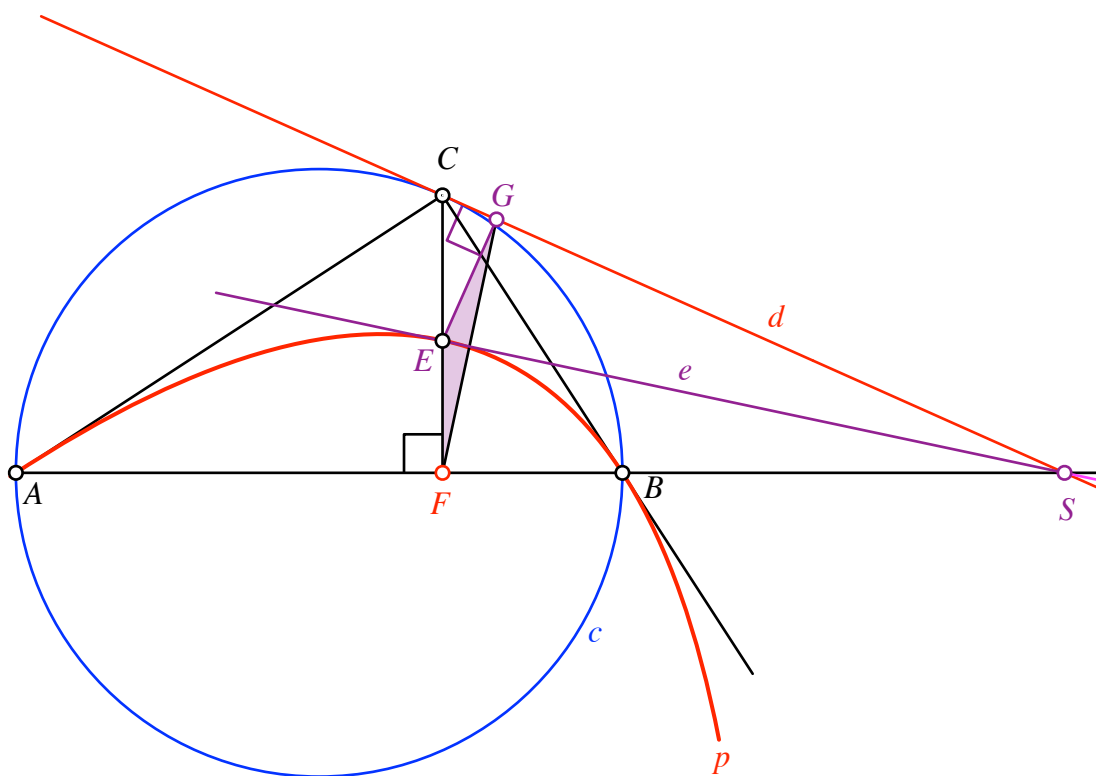
### 5.1 Parallel to the hypotenuse



**Fig. 6: Tangent parallel to the hypotenuse**

The midpoint  $D$  of the segment  $MC$  is on the parabola  $p$ . The tangent in  $D$  at the parabola  $p$  is parallel to the hypotenuse  $AB$ . The proof is given by the isosceles triangle  $FCD$ .

### 5.2 Point of intersection



**Fig. 7: Intersection of three lines**

Let  $S$  be the point of intersection of the lines  $d$  and  $AB$ , and  $E$  the common point of the parabola  $p$  and the line  $FC$ . The tangent  $e$  at  $p$  through  $E$  passes also through  $S$ . The proof derives from the isosceles triangle  $FGE$ .