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Means, Pythagoras, and Golden Section

Idea: M. N. D., N.

1 The Problem

Let p and q be two real numbers with $0 < q < p$ such that the arithmetic mean, the geometric mean and the harmonic mean of p and q are sides of a right triangle.

2 Solution

Since the arithmetic mean is the largest of the three means, we get by the theorem of Pythagoras:

$$\left(\tfrac{p+q}{2}\right)^2 = \sqrt{pq}^2 + \left(\tfrac{2pq}{p+q}\right)^2$$

We introduce the notation $\mu = \left(\frac{p}{q}\right)^2$. In this notation we have:

$$\left(\tfrac{1}{2}q\left(1+\sqrt{\mu}\right)\right)^2 = q^2\sqrt{\mu} + \frac{4q^2\mu}{\left(1+\sqrt{\mu}\right)^2},$$

or

$$\left(1+\sqrt{\mu}\right)^4 = 4\sqrt{\mu}\left(1+\sqrt{\mu}\right)^2 + 16\mu,$$

or

$$1+4\sqrt{\mu}+6\mu+4\sqrt{\mu}^3+\mu^2 = 4\sqrt{\mu}+8\mu+4\sqrt{\mu}^3+16\mu,$$

or

$$\mu^2 - 18\mu + 1 = 0.$$

Hence

$$\mu_{1,2} = 9 \pm 4\sqrt{5} = \left(2 \pm \sqrt{5}\right)^2.$$

Since $\mu > 1$ we have $\mu = \left(2+\sqrt{5}\right)^2 = (1+2\Phi)^2$, where $\Phi = \frac{1+\sqrt{5}}{2}$ (Golden Section, Walser 2001 and Walser 2013)).

Finally we get the condition $p = (1+2\Phi)q$.

3 Examples

3.1 The shape of the triangle

For $q=1$ we have the three sides:

$$\frac{p+q}{2}=\Phi^2$$

$$\sqrt{pq}=\Phi^{\frac{3}{2}}$$

$$\frac{2pq}{p+q}=\Phi$$

Figure 1 shows this example.

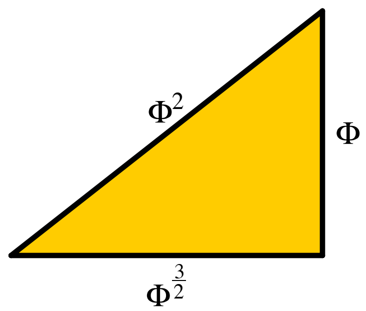

Fig. 1: The Right Golden Triangle

3.2 In the cardioid

The cardioid (Fig. 2) is a plane curve traced by a red point on the perimeter of the yellow circle (diameter 1) that is rolling around the fixed green circle of the same diameter.

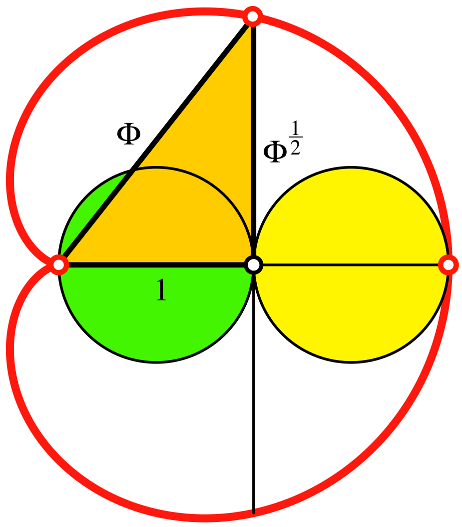

Fig. 2: In the cardioid

Inside the cardioid we find a Right Golden Triangle of the same shape (W1).

References

Walser, Hans (2001): *The Golden Section*. Translated by Peter Hilton and Jean Pedersen. The Mathematical Association of America 2001. ISBN 0-88385-534-8.

Walser, Hans (6. Auflage). (2013). *Der Goldene Schnitt*. Mit einem Beitrag von Hans Wußing über populärwissenschaftliche Mathematikliteratur aus Leipzig. Leipzig: Edition am Gutenbergplatz. ISBN 978-3-937219-85-1.

Links

W1: http://www.walser-h-m.ch/hans/Miniaturen/K/Kardioide/Kardioide.htm