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Fibonacci triangle

1 The triangle

Figure 1 gives the Fibonacci triangle.

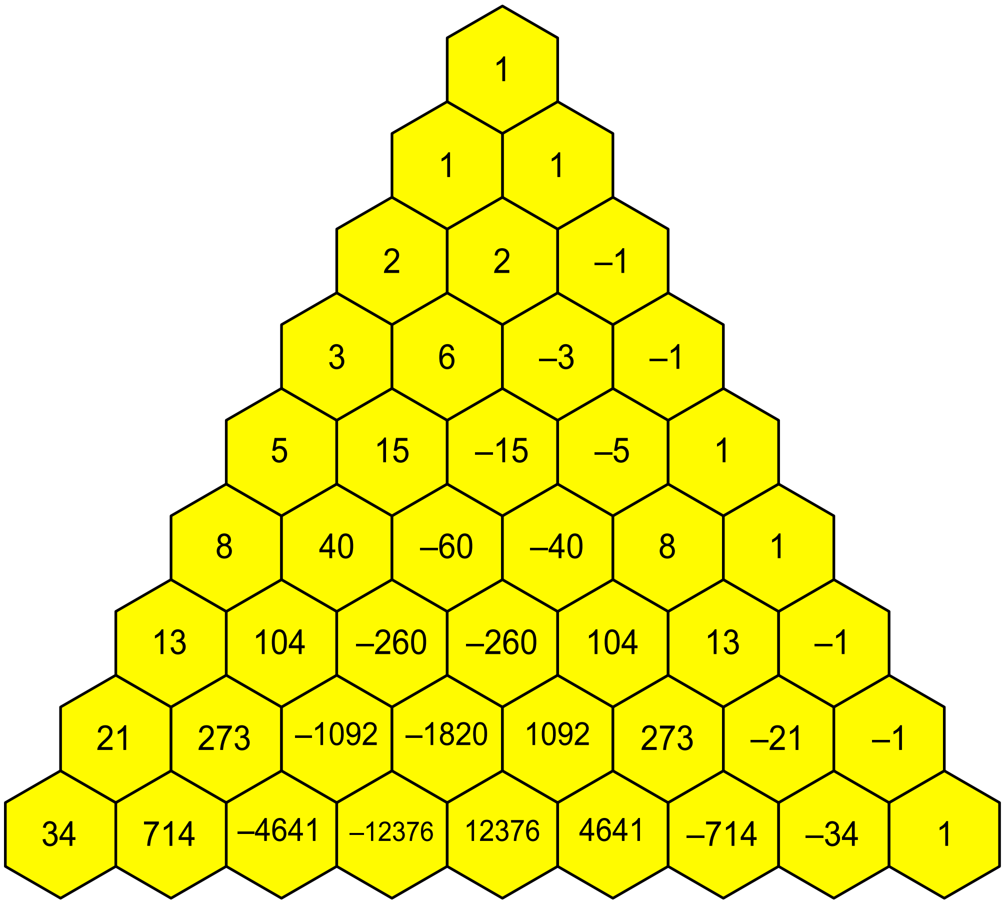

Fig. 1: Fibonacci triangle

In Figure 2 we see an orthogonal arrangement.

1								
1	1							
2	2	–1						
3	6	–3	–1					
5	15	–15	–5	1				
8	40	–60	–40	8	1			
13	104	–260	–260	104	13	–1		
21	273	–1092	–1820	1092	273	–21	–1	
34	714	–4641	–12376	12376	4641	–714	–34	1

Fig. 2: Orthogonal arrangement

2 Notation

For the Fibonacci numbers we use the usual upper case notation:

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13, F_8 = 21, F_9 = 34, \ldots \qquad (1)$$

For the Fibonacci triangle we will use the lower case notation of Figure 3.

$f_{0,0}$								
$f_{1,0}$	$f_{1,1}$							
$f_{2,0}$	$f_{2,1}$	$f_{2,2}$						
$f_{3,0}$	$f_{3,1}$	$f_{3,2}$	$f_{3,3}$					
$f_{4,0}$	$f_{4,1}$	$f_{4,2}$	$f_{4,3}$	$f_{4,4}$				
$f_{5,0}$	$f_{5,1}$	$f_{5,2}$	$f_{5,3}$	$f_{5,4}$	$f_{5,5}$			
$f_{6,0}$	$f_{6,1}$	$f_{6,2}$	$f_{6,3}$	$f_{6,4}$	$f_{6,5}$	$f_{6,6}$		
$f_{7,0}$	$f_{7,1}$	$f_{7,2}$	$f_{7,3}$	$f_{7,4}$	$f_{7,5}$	$f_{7,6}$	$f_{7,7}$	
$f_{8,0}$	$f_{8,1}$	$f_{8,2}$	$f_{8,3}$	$f_{8,4}$	$f_{8,5}$	$f_{8,6}$	$f_{8,7}$	$f_{8,8}$

Fig. 3: Notation

3 Features of the Fibonacci triangle

3.1 The Fibonacci numbers

The numbers in the first column of Figure 2 are the usual Fibonacci numbers:

$$f_{n,0} = F_{n+1} \qquad (2)$$

The Fibonacci numbers appear also under the roof:

$$f_{n,n-1} = (-1)^{\left\lceil \frac{n}{2} \right\rceil} F_{n+1} \qquad (3)$$

3.2 Signs

The signs change every second step (Fig. 4).

1								
1	1							
2	2	–1						
3	6	–3	–1					
5	15	–15	–5	1				
8	40	–60	–40	8	1			
13	104	–260	–260	104	13	–1		
21	273	–1092	–1820	1092	273	–21	–1	
34	714	–4641	–12376	12376	4641	–714	–34	1

Fig. 4: Signs

3.3 Symmetry

The Fibonacci triangle is not symmetric. But leaving away the “roof” on the right and ignoring the signs there is an axial symmetry (Fig. 5).

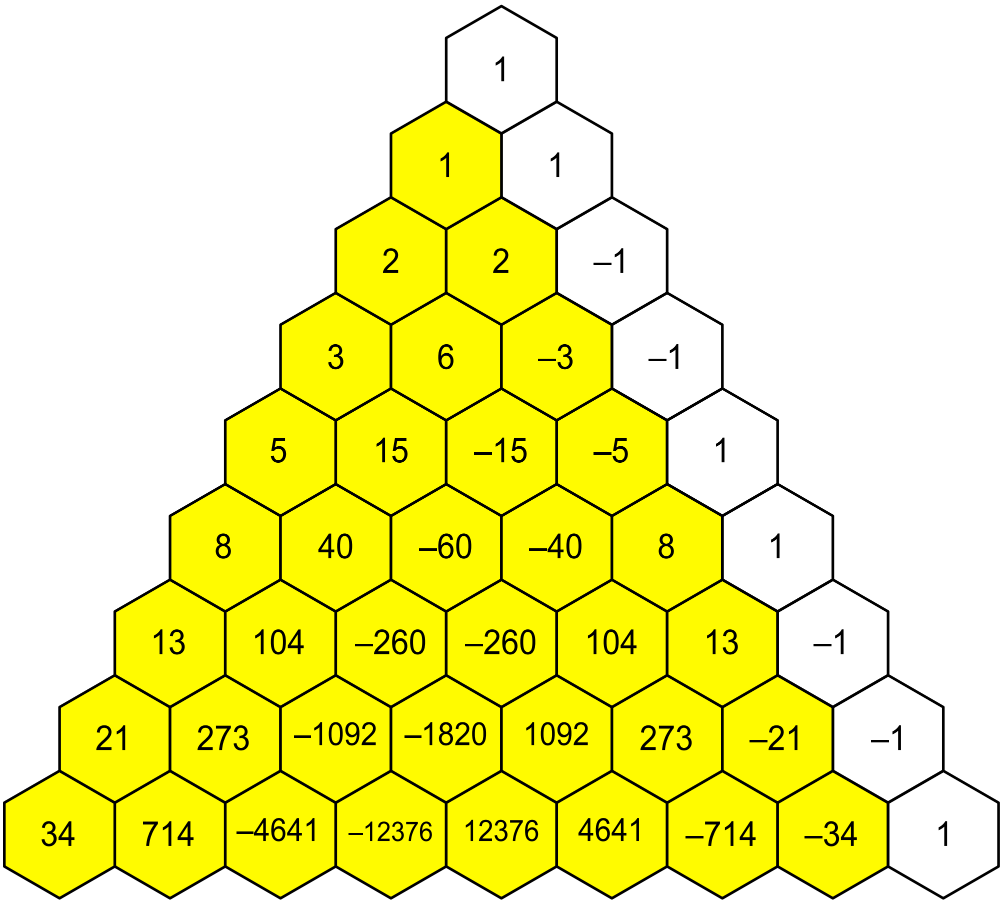

Fig. 5: Quasi symmetry

4 Powers of the Fibonacci numbers

4.1 Squares and cubes

Lets begin with a common picture (Fig. 6). It depicts the growth of the Fibonacci numbers. Beginning with a unit square we get the Fibonacci numbers for the side lengths of the square sequence. And of course we have the recursion:

$$F_{n+1} = F_n + F_{n-1} \tag{4}$$

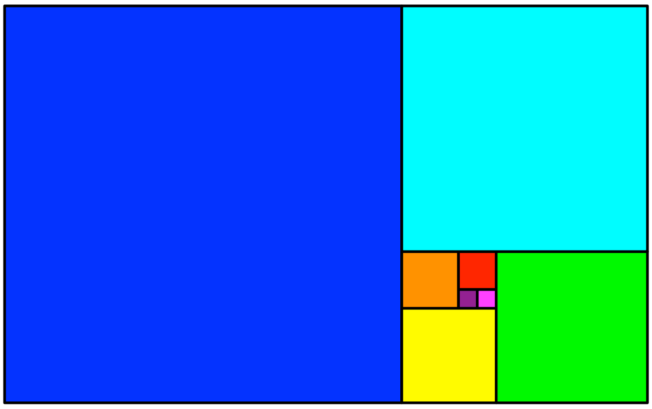

Fig. 6: Fibonacci squares

For the areas of the squares we obtain the sequence F_n^2:

$$1, 1, 4, 9, 25, 64, 169, 441, 1156, \ldots \tag{5}$$

Playing with these numbers we find a three digits recursion:

$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \tag{6}$$

Example:

$$441 = 2 \cdot 169 + 2 \cdot 64 - 25 \tag{7}$$

That means, that in Figure 6 the green and the cyan square have together half the area of the yellow plus the blue square. This can be seen by the dissection proof of Figure 7.

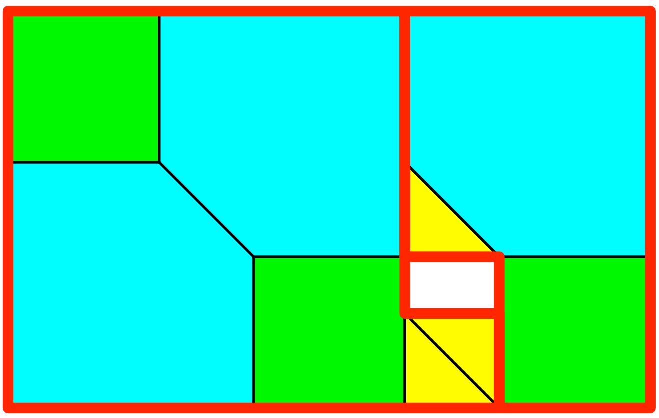

Fig. 7: Dissection

Notice that the coefficients of (6) appear in the Fibonacci triangle:

$$F_{n+1}^2 = f_{2,0}F_n^2 + f_{2,1}F_{n-1}^2 + f_{2,2}F_{n-2}^2 \tag{8}$$

For the cubes F_n^3 of the Fibonacci numbers we have:

$$1, 2, 8, 27, 125, 512, 2197, 9261, 39304, \ldots \tag{9}$$

Here we find a four digits recursion:

$$F_{n+1}^3 = f_{3,0}F_n^3 + f_{3,1}F_{n-1}^3 + f_{3,2}F_{n-2}^3 + f_{3,3}F_{n-3}^3 \tag{10}$$

Example:

$$2197 = 3 \cdot 512 + 6 \cdot 125 + (-3) \cdot 27 + (-1) \cdot 8 \tag{11}$$

4.2 General powers

For the *k*th power of the Fibonacci numbers there is a (k + 1) digits recursion:

$$F_{n+1}^k = f_{k,0}F_n^k + f_{k,1}F_{n-1}^k + \cdots + f_{k,k}F_{n-k}^k = \sum_{j=0}^{k} f_{k,j}F_{n-j}^k \tag{12}$$

For the proof of (12) I used a computer algebra system.

5 The elements of the Fibonacci triangle

5.1 Columns

We have already found (2) the Fibonacci numbers in the first column of Figure 2.

In the next column of Figure 2 we have the products of two consecutive Fibonacci numbers:

$$f_{n,1} = F_{n+1}F_n \tag{13}$$

Example:

$$15 = f_{4,1} = F_5F_4 = 5 \cdot 3 \tag{14}$$

Again in the next column we get products of three consecutive Fibonacci numbers, but with a negative sign an a coefficient of one half:

$$f_{n,2} = -\tfrac{1}{2} F_{n+1} F_n F_{n-1} \tag{15}$$

Example:

$$-1092 = f_{7,2} = -\tfrac{1}{2} F_8 F_7 F_6 = -\tfrac{1}{2} \cdot 21 \cdot 13 \cdot 8 \tag{16}$$

5.2 Overview

For the first five columns we get:

$f_{n,0} = F_{n+1}$
$f_{n,1} = F_{n+1} F_n$
$f_{n,2} = -\tfrac{1}{2} F_{n+1} F_n F_{n-1}$
$f_{n,3} = -\tfrac{1}{6} F_{n+1} F_n F_{n-1} F_{n-2}$
$f_{n,4} = \tfrac{1}{30} F_{n+1} F_n F_{n-1} F_{n-2} F_{n-3}$

The denominators in the coefficients are the products of consecutive Fibonacci numbers beginning with F_1 (kind of “factorial”):

$$\begin{aligned} \tfrac{1}{2} &= \tfrac{1}{F_1 F_2 F_3} \\ \tfrac{1}{6} &= \tfrac{1}{F_1 F_2 F_3 F_4} \\ \tfrac{1}{30} &= \tfrac{1}{F_1 F_2 F_3 F_4 F_5} \end{aligned} \tag{17}$$

5.3 General

Finally we get the general formula for the elements of the Fibonacci triangle:

$$f_{n,k} = (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{\prod_{j=0}^{k} F_{n+1-j}}{\prod_{j=0}^{k} F_{j+1}} \tag{18}$$

6 Recursion in the Columns

The elements in the columns of the Fibonacci triangle fulfill a recursion like the recursion (12) for the powers of the Fibonacci numbers. Id est:

$$f_{n,k} = \sum_{j=0}^{k+1} f_{k+1,j}\, f_{n-1-j,k} \tag{19}$$

Examples:
First example: $n = 7, k = 1$

$$\begin{aligned} f_{7,1} &= f_{2,0}\, f_{6,1} + f_{2,1}\, f_{5,1} + f_{2,2}\, f_{4,1} \\ 273 &= \ 2 \cdot 104 + \ 2 \cdot 40 \ + (-1) \cdot 15 \end{aligned} \tag{20}$$

Figure 8 shows the involved elements. Red is the dot product of cyan and yellow.

1								
1	1							
2	2	–1						
3	6	–3	–1					
5	15	–15	–5	1				
8	40	–60	–40	8	1			
13	104	–260	–260	104	13	–1		
21	273	–1092	–1820	1092	273	–21	–1	
34	714	–4641	–12376	12376	4641	–714	–34	1

Fig. 8: Dot product

Second example: $n = 8, k = 2$

$$f_{8,2} = f_{3,0}\, f_{7,2} + f_{3,1}\, f_{6,2} + f_{3,2}\, f_{5,2} + f_{3,3}\, f_{4,2}$$
$$-4641 = 3\cdot(-1092) + 6\cdot(-260) + (-3)\cdot(-60) + (-1)\cdot(-15)$$

In Figure 9 the involved elements:

1								
1	1							
2	2	–1						
3	6	–3	–1					
5	15	–15	–5	1				
8	40	–60	–40	8	1			
13	104	–260	–260	104	13	–1		
21	273	–1092	–1820	1092	273	–21	–1	
34	714	–4641	–12376	12376	4641	–714	–34	1

Fig. 9: Involved elements